

# Some four-loop corrections to the anomalous magnetic moment of the muon

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# Outline

1 Introduction

2 Hadronic contribution

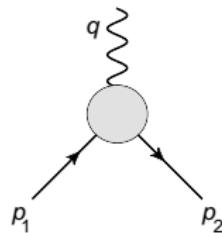
3 Heavy-lepton correction

# Anomalous magnetic moment

$$\vec{\mu} = g \frac{e}{2m} \vec{s} \quad g = 2 \cdot (1 + a)$$

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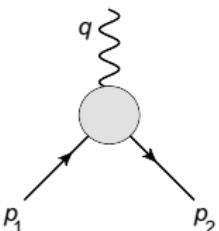


$$= -ie \bar{\psi}(p_2) \left( \gamma^\mu F_E(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2m} F_M(q^2) \right) \psi(p_1)$$

$$F_E(0) = 1, \quad F_M(0) = a$$

# Anomalous magnetic moment

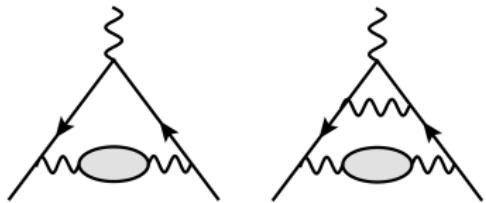
$$\vec{\mu} = g \frac{e}{2m} \vec{s} \quad g = 2 \cdot (1 + a)$$


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[PDG]

$$a_\mu^{\text{exp}} = 11659208.9(6.3) \cdot 10^{-10}$$
$$a_\mu^{\text{th}} = 11659180.3(4.9) \cdot 10^{-10} \Rightarrow \Delta a_\mu \approx 3 \sigma$$

# Hadronic contribution



LO

[Davier et al 2010] [Hagiwara et al 2011]

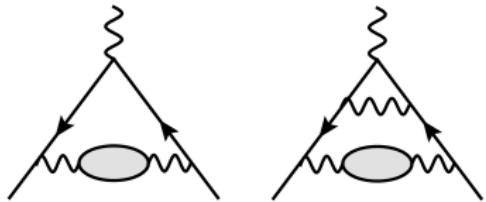
[Jegerlehner et al 2011] [Benayoun et al 2012]

NLO

[Barbieri, Remiddi 1975] [Krause 1997]

NNLO

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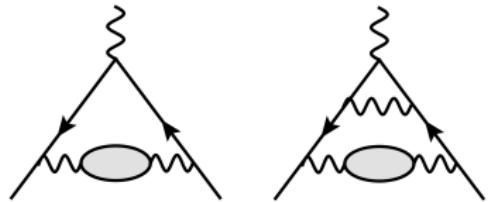
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NNLO

$$a_\mu^{(1)} = \int d\ell \left[ P^\omega e \gamma_\rho \frac{1}{\not{p} + \not{q} + \not{\ell} - m_\mu} \gamma_\omega \frac{1}{\not{p} + \not{\ell} - m_\mu} e \gamma_\lambda \frac{g^{\rho\mu}}{\ell^2} \frac{g^{\nu\lambda}}{\ell^2} \right] (\ell^2 g_{\mu\nu} - \ell_\mu \ell_\nu) \Pi(\ell^2)$$

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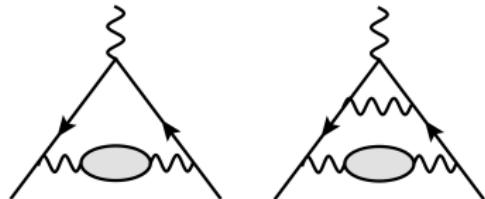
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$$= \int d\ell [\dots] (\ell^2 g_{\mu\nu} - \ell_\mu \ell_\nu) \cdot \frac{\ell^2}{\pi} \int_{m_\pi^2}^\infty \frac{1}{\ell^2 - s} \text{Im} \Pi(s) \frac{ds}{s}$$

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NNLO

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# Hadronic calculation

$$a_\mu^{(3)} = \frac{1}{3} \left(\frac{\alpha}{\pi}\right)^4 \int_{m_\pi^2}^\infty ds K^{(3)}(s) R(s) / s$$



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compilation from [Nomura, Teubner]
- resonances  $J/\Psi$ ,  $\Psi'$  and  $\Upsilon(nS)$   
in narrow-width approximation

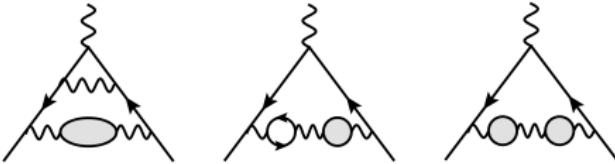


$$R(s) \sim \Gamma_{ee} M_R \cdot \delta(s - M_R^2)$$

# Hadronic result

$$a_{\mu}^{\text{NLO}} \cdot 10^{10} = -20.90 + 10.68 + 0.35 = -9.87(9)$$

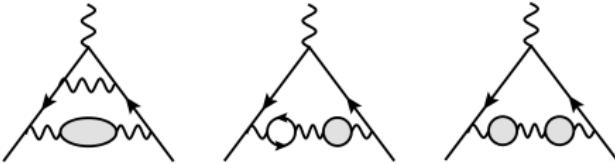
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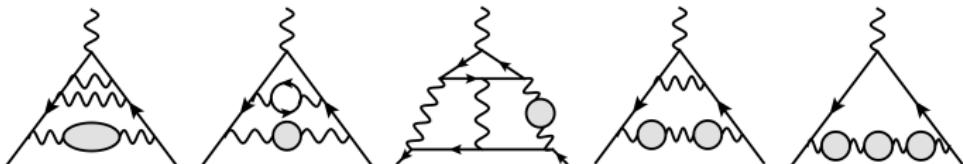
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$$a_{\mu}^{\text{NNLO}} \cdot 10^{10} = 0.80 - 0.41 + 0.91 - 0.06 + 0.0005 = 1.24(1)$$

[Kurz, Liu, Marquard, Steinhauser 2014]



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$$\Delta a_{\mu}^{\text{old}} \Rightarrow \Delta a_{\mu}^{\text{new}} = a_{\mu}^{\text{exp}} - a_{\mu}^{\text{th}} - a_{\mu}^{\text{NNLO}}$$
$$= 23.7(8.6) \cdot 10^{-10}$$

$$2.9 \sigma \Rightarrow 2.7 \sigma$$

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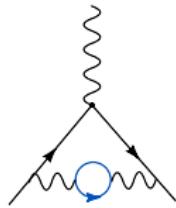
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$a_\mu^{\text{HLbL,NLO}} = 0.3(2) \cdot 10^{-10}$

[Colangelo, Hoferichter, Nyffeler, Passera, Stoffer 2014]

# Heavy-lepton correction



$$a_\mu(\tau) \sim \mathcal{O}\left(\frac{m_\mu^2}{m_\tau^2}\right)$$

$$a_\mu(e) \sim \mathcal{O}\left(\ln \frac{m_\mu}{m_e}\right)$$

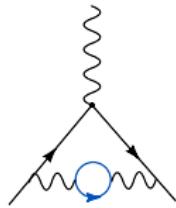
$2\ell$  [Eliel 1966]

$3\ell$  [Laporta, Remiddi 1993; Laporta 1993]

$4\ell$  [Kinoshita, Nio 2003] [Lee et al 2013]

$5\ell$  [Aoyama et al 2011]

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## Calculation:

- QGRAF: generate Feynman diagrams [Nogueira]
- q2e/exp: asymptotic expansion with  $m_\tau \gg m_\mu \gg m_e$   
[Harlander, Seidensticker, Steinhauser]
- FORM: treatment of complicated structure [Vermaseren]
- FIRE: reduction to master integrals [A.V.Smirnov]
- renormalization in on-shell scheme

## Heavy-lepton result

$$a_\mu^{\text{QED}} = a_{\text{uni}} + a_\mu(\tau) + a_\mu(e) + a_\mu(\tau, e)$$

$$a_\mu(\tau) = 0.0078 \cdot 10^{-2} \cdot \left(\frac{\alpha}{\pi}\right)^2 + 0.0361 \cdot 10^{-2} \cdot \left(\frac{\alpha}{\pi}\right)^3 + A_\mu^{(8)}(\tau) \cdot \left(\frac{\alpha}{\pi}\right)^4$$

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$$\begin{aligned} A_\mu^{(8)}(\tau) &= \left(\frac{m_\mu}{m_\tau}\right)^2 \left( \frac{4\pi^2\zeta_3}{15} - \frac{52\ln^5(2)}{675} - \frac{3851\pi^2}{3600} + \dots \right. \\ &\quad \left. + \ln \frac{m_\mu^2}{m_\tau^2} \left( -\frac{38891}{12150} + \frac{19\pi^2}{135} + \frac{3\zeta_3}{2} \right) + \frac{359}{1080} \ln^2 \frac{m_\mu^2}{m_\tau^2} \right) \\ &\quad + \left(\frac{m_\mu}{m_\tau}\right)^3 \frac{\pi^2}{90} + \dots + \mathcal{O}\left(\left(\frac{m_\mu}{m_\tau}\right)^8\right) \end{aligned}$$

$$\approx 4.24941(2)(53) \cdot 10^{-2} \quad \leftrightarrow \quad 4.234(12) \cdot 10^{-2}$$

[Kurz, Liu, Marquard, Steinhauser 2014]

[Aoyama, Hayakawa, Kinoshita, Nio 2012]

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$2\ell$        $3\ell$        $4\ell$

$$a_\mu(\tau) = (4.213 + 0.045 + 0.012) \cdot 10^{-10} \quad \leftrightarrow \quad a_{\text{uni}}^{5\ell} = 0.006 \cdot 10^{-10}$$

## Conclusions

- Asymptotic calculation of  $a_\mu(\tau)$  at  $\mathcal{O}(\alpha^4)$   
→ numerical agreement with literature
- New result for  $a_\mu^{\text{had}}$  at NNLO  
⇒ significant change of  $\Delta a_\mu$  by  $0.2 \sigma$
- Work on  $a_\mu(e)$  in progress

# BACKUP

diagram group	$10^2 \cdot A_\mu^{(8)}(\tau)$	
	our work	ref.
I(a)	0.00324281(2)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	-0.6292808(6)	-0.6293(1)
I(d)	0.0367796(4)	0.0368(0)
III	4.5208986(6)	4.504(14)
II(a) + IV(d)	-2.316756(5)	-2.3197(37)
IV(a)	3.851967(3)	3.8513(11)
IV(b)*	0.612661(5)	0.6106(31)
IV(c)	-1.83010(1)	-1.823(11)

\* analytical check of order  $\left(\frac{m_\mu}{m_\tau}\right)^2$  [Boughezal, Melnikov 2011] [Kataev 2012]

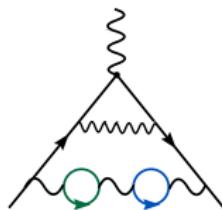
## $g - 2$ of electron

$$a_e^{\text{QED}} = a_{\text{uni}} + a_e(\mu) + a_e(\tau) + a_e(\mu, \tau)$$

$$\begin{aligned} a_e^{(8)}(\mu) &= 9.161970703(2)(372) \cdot 10^{-4} \cdot \left(\frac{\alpha}{\pi}\right)^4 \leftrightarrow 9.222(66) \cdot 10^{-4} \cdot \left(\frac{\alpha}{\pi}\right)^4 \\ &= 2.7 \cdot 10^{-14} \approx \delta a_e^{\text{exp}} / 10 \end{aligned}$$

[Aoyama, Hayakawa, Kinoshita, Nio 2012]

$$A_e^{(8)}(\tau) = 7.42924(0)(118) \cdot 10^{-6} \leftrightarrow 7.38(12) \cdot 10^{-6}$$



$$A_e^{(8)}(\mu, \tau) = \frac{m_e^2}{m_\tau^2} \left( \frac{89}{810} \ln^2 \frac{m_\mu^2}{m_\tau^2} + \dots \right) + \mathcal{O}\left(\frac{1}{M^6}\right)$$

$$\approx 7.4687(26)(10) \cdot 10^{-7} \leftrightarrow 7.465(18) \cdot 10^{-7}$$

[Aoyama, Hayakawa, Kinoshita, Nio 2012]



$$a_e^{\text{had,NNLO}} = 2.8(1) \cdot 10^{-14}$$